

Topological order and the vacuum of Yang-Mills theories.

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We study, for $SU(2)$ Yang-Mills theories discretized on a lattice, a non-local topological order parameter, the center flux z . We show that: i) well defined topological sectors classified by $\pi_1(SO(3)) = \mathbb{Z}_2$ can only exist in the ordered phase of z ; ii) depending on the dimension $2 \leq d \leq 4$ and action chosen, the center flux exhibits a critical behaviour sharing striking features with the Kosterlitz-Thouless type of transitions, although belonging to a novel universality class; iii) such critical behaviour does not depend on the temperature T . Yang-Mills theories can thus exist in two different continuum phases, characterized by an either topologically ordered or disordered vacuum; this reminds of a quantum phase transition, albeit controlled by the choice of symmetries and not by a physical parameter.

I. INTRODUCTION

Of all ideas applied to the confinement problem in non-abelian Yang-Mills theories [1, 2] the most popular still involve topological degrees of freedom of some sort [3–11]. Among these center vortices [12–14] have enjoyed broad attention, in particular in the lattice literature. Although most of the effort was put in dealing with gauge fixed schemes,¹ some investigations actually attempted to tackle the problem in a gauge invariant way [17–23] and are therefore directly related to 't Hooft's original idea.

Non-abelian gauge fields transform under the group's adjoint representation, $SU(N)/\mathbb{Z}_N$. Such group is not simply connected, with a non trivial first homotopy class:

$$\pi_1\left(\frac{SU(N)}{\mathbb{Z}_N}\right) = \mathbb{Z}_N, \quad (1)$$

the center of $SU(N)$. Following Ref. [13], let us consider the Euclidean Yang-Mills theory on a d -dimensional torus,² i.e. with all directions compactified, and choose one of the d Euclidean directions as time. If large gauge transformations classified by Eq. (1) induce a super-selection rule, we can decompose the physical Hilbert space \mathcal{H} of gauge invariant states [25] in sub-spaces $\mathcal{H}_{\vec{k}, \vec{m}}$ labeled by topological indices, the \mathbb{Z}_N electric and magnetic fluxes (vortices) $\vec{k} = (k_1, \dots, k_{n_t})$ and $\vec{m} = (m_1, \dots, m_{n_s})$ [13]:

$$\mathcal{H} = \bigoplus_{k_i, m_j=0}^{N-1} \mathcal{H}_{\vec{k}, \vec{m}}. \quad (2)$$

Here $n_t = d - 1$ counts the space-time and $n_s = \frac{(d-1)(d-2)}{2}$ the space-space planes and $k_i, m_i \in \mathbb{Z}_N$. As 't Hooft pointed out, a sufficient condition for confinement is realized if the low-temperature phase of pure Yang-Mills theories corresponds to a superposition of all (electric) sectors, while above the deconfinement transition such \mathbb{Z}_N symmetry must get broken to the trivial one [12, 13].³

One can check such scenario by calculating, e.g. in lattice simulations, how the free energy for \mathbb{Z}_N flux creation:

$$F(\vec{k}) = \Delta \mathcal{U}_{\vec{k}} - T \Delta \mathcal{S}_{\vec{k}} = -\log \frac{Z(\vec{k})}{Z(\vec{0})} \quad (3)$$

¹ See e.g. Refs. [15, 16] for early results and Ref. [2], Chaps. 6, 7 for a comprehensive review. The goal here is to isolate some relevant degrees of freedom, usually called P-vortices, assumed to be related to 't Hooft's topological excitations; we will comment on this in Sec. IV.

² See e.g. Ref. [24] for an extensive introduction to the subject.

³ Magnetic sectors, on the other hand, can remain unbroken and be responsible for screening effects.

changes with the temperature T across the deconfinement transition. Here $\Delta\mathcal{U}_{\vec{k}}$ is the energy (action) cost to generate the \vec{k}^{th} (electric) vortex from the vacuum, $\Delta\mathcal{S}_{\vec{k}}$ the corresponding entropy change and $Z(\vec{k})$ the partition function restricted to the topological sector labeled by \vec{k} .⁴ For the one-vortex sector, F is nothing but the free energy of a maximal 't Hooft loop,⁵ giving a confinement criterion dual to Wilson's: in the thermodynamic limit F should vanish in the confined phase while it should diverge as $\tilde{\sigma}(T)L^2$ above the deconfinement temperature T_c , where $\tilde{\sigma}(T)$ is the dual string tension [2, 13, 17–21, 27]. In other words, a perimeter law for the Wilson loop implies an area law for the 't Hooft loop and vice-versa [12, 13].

Of course, when considering the theory at $T = 0$, the distinction between electric and magnetic fluxes is artificial. In this case *all* the $N^{\frac{d(d-1)}{2}}$ topological sectors must be taken into account when establishing whether \mathbb{Z}_N -symmetry is unbroken, i.e. whether the vacuum $|\Psi_0\rangle$ is indeed a symmetric superposition of states belonging to $\mathcal{H}_{\vec{k},\vec{m}}$.⁶

$$|\Psi_0\rangle = \sum_{\vec{k},\vec{m}} |\Psi_0^{\vec{k},\vec{m}}\rangle. \quad (4)$$

In the following we will use either definition, depending on whether we are considering the $T = 0$ or the $T > 0$ case.

The above ideas generalize naturally to the lattice discretization of Yang-Mills theories; the specific action used plays however a key role in their actual implementation. If one wishes to preserve the symmetries of the continuum theory the natural choice should fall on a discretization transforming under the “correct” group $SU(N)/\mathbb{Z}_N$. One possibility among many (see e.g. Ref. [28]) is given by the adjoint Wilson action with periodic boundary conditions [27]:

$$S_A = \beta_A \sum_P \left(1 - \frac{1}{N^2 - 1} \text{Tr}_A U_P \right), \quad (5)$$

where U_P is the standard plaquette. For $N = 2$ it was indeed shown in Refs. [21, 22, 29] that for simulations based on Eq. (5):

i) \mathbb{Z}_2 topological sectors are well defined in the continuum limit, both below and above T_c , i.e. the decomposition in Eq. (2) holds;

⁴ In the deconfined phase all electric sectors must be suppressed relatively to the trivial one, while in the confined phase all \vec{k} and \vec{m} should be equally probable.

⁵ For a representation of the 't Hooft loop in the continuum see Ref. [26]

⁶ Actually, in virtue of cubic symmetry, one can regard the sub-spaces $\mathcal{H}_{\vec{k},\vec{m}}$ with indices equal up to a permutation as equivalent and recombine them in Eq. (4) into weights given by their combinatorial multiplicity [17–19].

- ii) the partition function $Z_A = \int \exp(-S_A)$ dynamically includes all sectors;
- iii) in the deconfined phase all non trivial sectors are suppressed, while as $T \rightarrow 0$ all sectors are equivalent, i.e. the vacuum can be described by Eq. (4).

The main difficulty of such setup lies of course in the implementation of an algorithm capable of tunneling ergodically among all vortex topologies. Simulations are therefore quite demanding: reaching enough statistics to check whether the symmetry among sectors postulated in Eq. (2) remains unbroken from $T = 0$ all the way up to T_c is difficult; the evidence given in Refs. [21, 22] seems to point to a more complicated picture.

Alternatively, universality [30] should allow the use of the fundamental Wilson action:

$$S_F = \beta_F \sum_P \left(1 - \frac{1}{N} \Re [\text{Tr}_F U_P] \right), \quad (6)$$

which is the quenched (mass $\rightarrow \infty$) limit of the physical action coupling Yang-Mills theories to fundamental fermions, e.g. full QCD. In this case, however, some care must be taken in defining a $SU(N)/\mathbb{Z}_N$ invariant theory. Indeed, in the presence of fundamental fermions the topological classification of Eq. (1) breaks down.⁷ The extension to full QCD has been indeed one of the main obstacles in establishing the 't Hooft vortex picture as a viable model for confinement. We will comment on this in Sec. IV; for the moment, let us note that one can still introduce vortex topological sectors “statically” by simply imposing twisted boundary conditions [19, 20, 27].⁸ We should then be able to reconstruct the “full” partition function Z_F by taking the weighted sum of all partition functions $Z_F(\vec{k}, \vec{m}) = \int \exp(-S_F(\vec{k}, \vec{m}))$ with boundary conditions corresponding to the sector labeled by \vec{k} and \vec{m} [19, 20]. Since each $Z_F(\vec{k}, \vec{m})$ must be determined via independent simulations, their relative weights can only be calculated through indirect means.⁹ Still, such simulations are computationally more efficient than in the Z_A case and have therefore been the method of choice in most investigations of Eq. (3) [17–20, 23].

Investigations using Eq. (6) rely on the assumption that fixing the boundary conditions is enough to ensure that the Hilbert-space decomposition defined in Eq. (2) works. However, it is well known that upon discretization of Yang-Mills theories \mathbb{Z}_N magnetic monopoles are generated at strong coupling [28, 35–39], causing bulk phenomena in the $\beta_F - \beta_A$ phase

⁷ See e.g. Ref. [31] for a recent discussion.

⁸ Such topological boundary conditions, relevant e.g. in investigations of large N reduction [24, 32, 33], allow adjoint fermions but no fundamental ones. Flavour twisted boundary conditions, on the other hand, are well established in full QCD [34].

⁹ See e.g. Ref. [23], Chapt. 3 for a detailed review of the methods involved.

diagram. Now, since the \mathbb{Z}_N fluxes defining our topological sectors live on the co-set of a two dimensional plane, they have a simple geometrical interpretation: they are described in $d = 4$ by a *closed* world-sheet, i.e. they are string-like objects, and in $d = 3$ by a *closed* world-line, i.e. particle-like. On the other hand, topological lattice artifacts as the above mentioned \mathbb{Z}_N monopoles are themselves sources of \mathbb{Z}_N flux: in $d = 4$ they will be particle-like objects, their *closed* world-lines bounding *open* \mathbb{Z}_N flux world-sheets, while in $d = 3$ they will be instanton-like objects and will be end-points of *open* \mathbb{Z}_N flux lines [27, 35–37, 39]. \mathbb{Z}_N monopoles are therefore in one to one correspondence with *open* center vortices; in other words, universality between the fundamental and adjoint actions can only be invoked when just *closed*, i.e. truly topological \mathbb{Z}_N vortices winding around the compactified directions can form. Notice how in $d = 2$, where no \mathbb{Z}_N monopoles can exist, \mathbb{Z}_N fluxes are instanton type objects. The distinction between open and closed vortices is in this case blurred, but in a non-ergodic setup it can eventually be made through the flux allowed by the boundary conditions chosen.

The above discussion has a straightforward consequence. If one could “measure” whether open \mathbb{Z}_N vortices are absent in a given discretization, i.e. whether only topological vortices can be generated from the vacuum, there would be no need to monitor \mathbb{Z}_N monopoles to establish universality between S_A and S_F in the first place, since these must be absent anyway. This would have two advantages: first, such criterion could be generalized to $d = 2$. Second, absence of lattice artifacts, whether for S_F , for S_A or for both, would get “promoted” to a necessary condition for the super-selection rule of Eq. (2), and hence for the conjectured vacuum symmetry of Eq. (4), to be realized. Indeed, consider states belonging to distinct topological sectors labeled by the indices $k, k' \in \mathbb{Z}_N$. The presence of open vortices immediately blurs the distinction among them: does the state pictured at the top of Fig. 1 belong to the k^{th} sector, resulting from the superposition of k closed vortices with $\text{mod}(k' - k)_N$ open ones (middle picture), or does it belong to the k'^{th} sector, coming from the superposition of k' closed vortices with $\text{mod}(k - k')_N$ open ones winding in the other direction (bottom picture)? Clearly, there is no way to distinguish between them and assign the configuration to $Z(k)$ rather than $Z(k')$ in Eq. (3). In other words, a Wilson loop will never know if the vortex piercing it to generate the area law for its expectation value [12, 13] is open or closed: a confinement criterion based on the vortex free energy F and hence on the 't Hooft loop can only make sense if open vortices are absent at any temperature.

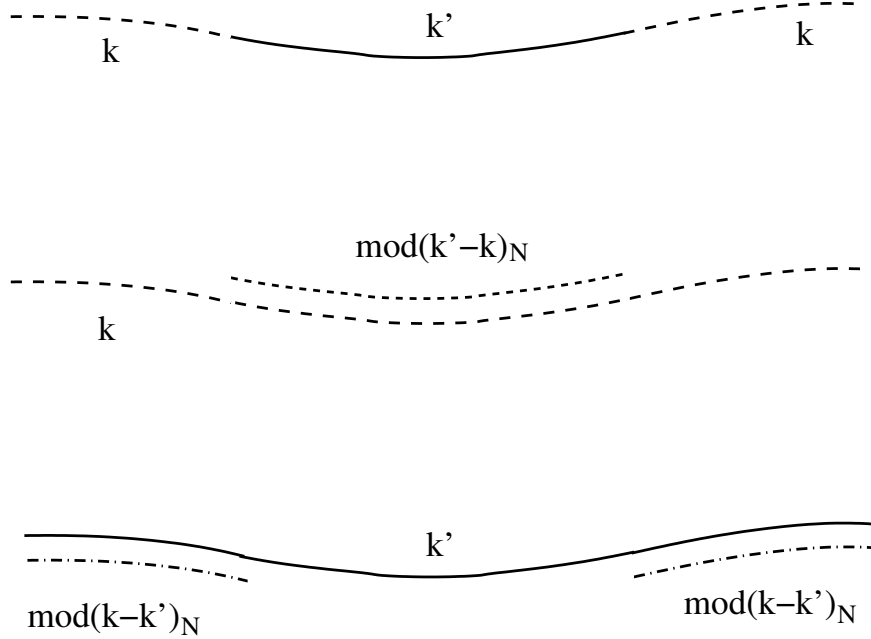


FIG. 1. Illustration of the ambiguity in labeling the \mathbb{Z}_N topological sectors in presence of open vortices. Should the given configuration (top) be counted to the k^{th} (middle) or to the k'^{th} (bottom) sector?

In this paper we will investigate a topological order parameter, the center flux z , for the transition between phases characterized by the presence of open or closed \mathbb{Z}_2 vortices in $SU(2)$ Yang-Mills theories at $T = 0$, discretized through standard plaquette actions. We will show that, depending on the action, the dimensions and the volume, the theory can be either in a topologically ordered or disordered phase; such distinction will persist at finite T . In the disordered phase open vortices dominate the vacuum and \mathbb{Z}_2 topological sectors are ill defined; the Hilbert space of Yang-Mills theories cannot be classified by a super-selection as in Eq. (2). Such disordered phase is compatible with the presence of fundamental fermions; the ordered phase, on the other hand, should be the correct one when coupling $SU(2)$ with adjoint fermions, a popular candidate for infrared conformal gauge theories.

Besides this (perhaps lengthy) introduction, the rest of the paper is organized as follows: Sec. II contains details on the lattice setup, observables and simulation techniques; in Sec. III the main results will be presented; Sec. IV contains the conclusions and outlook. Preliminary results of this investigations have been presented in Refs. [40, 41].

II. SETUP

A. Action and Observables

We will consider the $SU(2)$ mixed fundamental-adjoint Wilson action with periodic boundary conditions in $2 \leq d \leq 4$ Euclidean dimensions, as given in Eqs. (5, 6):

$$S = \beta_A \sum_P \left(1 - \frac{1}{3} \text{Tr}_A U_P \right) + \beta_F \sum_P \left(1 - \frac{1}{2} \text{Tr}_F U_P \right)$$

$$\frac{1}{a^{4-d} g^2} = \frac{1}{4} \beta_F + \frac{2}{3} \beta_A, \quad (7)$$

where a is the (dimensionful) lattice spacing and U_P denotes the 1×1 plaquette; all results can be easily generalized to different boundary conditions. For higher groups $SU(N)$ the general picture should not change dramatically [29, 42–46]. However, other representations than just the fundamental and adjoint are allowed. Many details might therefore depend on N ; direct investigations of at least the $SU(3)$ case would be welcome.

In $d \geq 3$, \mathbb{Z}_2 monopoles can be defined for each elementary cube c through the product:

$$\sigma_c = \prod_{P \in \partial c} \text{sign}(\text{Tr}_F U_P), \quad (8)$$

over all plaquettes U_P belonging to its surface ∂c [27–29, 38, 47]. Notice how rescaling any link by a \mathbb{Z}_2 factor will leave σ_c unchanged.

The \mathbb{Z}_2 monopole density should vanish in the continuum limit $g^2 \rightarrow 0$. This happens, however, in different ways, depending on the dimensions d or the direction along which such limit is taken in the $\beta_F - \beta_A$ plane, and has been the subject of intense investigations in the pioneering years of lattice gauge theories [38, 42–46, 48–54]. For the $SU(2) - SO(3)$ case considered here the resulting phase diagrams in $d = 3$ and 4 are sketched in Figs. 2, 3; similar ones have been established for $N \geq 3$, see e.g. Refs. [42–46]. Continuous lines indicate bulk transitions [38, 49, 50, 52], dashed lines the roughening transition [48], dotted lines the crossover regions associated with \mathbb{Z}_2 monopoles [38, 52]. The vertical bulk transition line coming down from $\beta_A = \infty$ corresponds to the underlying \mathbb{Z}_2 gauge theory: in $d = 3$ it ends at a finite point [52–54], while in $d = 4$ it joins the bulk transition line associated with \mathbb{Z}_2 monopoles [38, 49, 50]; from the endpoint of the latter a crossover region starts, extending beyond the β_F axis. In $d = 2$, \mathbb{Z}_2 monopoles are of course absent. Furthermore, the \mathbb{Z}_2 gauge

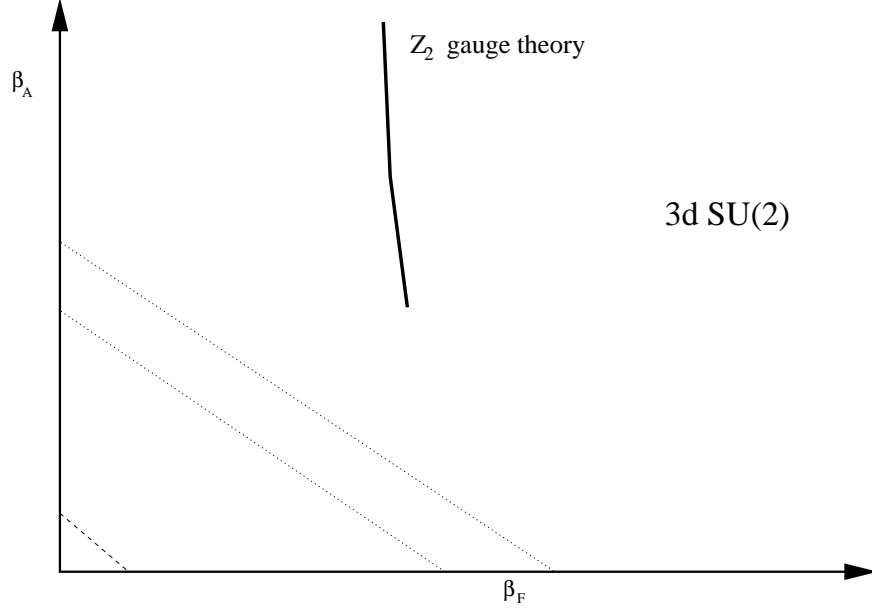


FIG. 2. Phase diagram of the fundamental-adjoint plane for $d = 3$. Continuous lines indicate bulk transitions, dashed lines the roughening transition, dotted lines the crossover regions associated with \mathbb{Z}_2 monopoles. Similar diagrams hold for higher N .

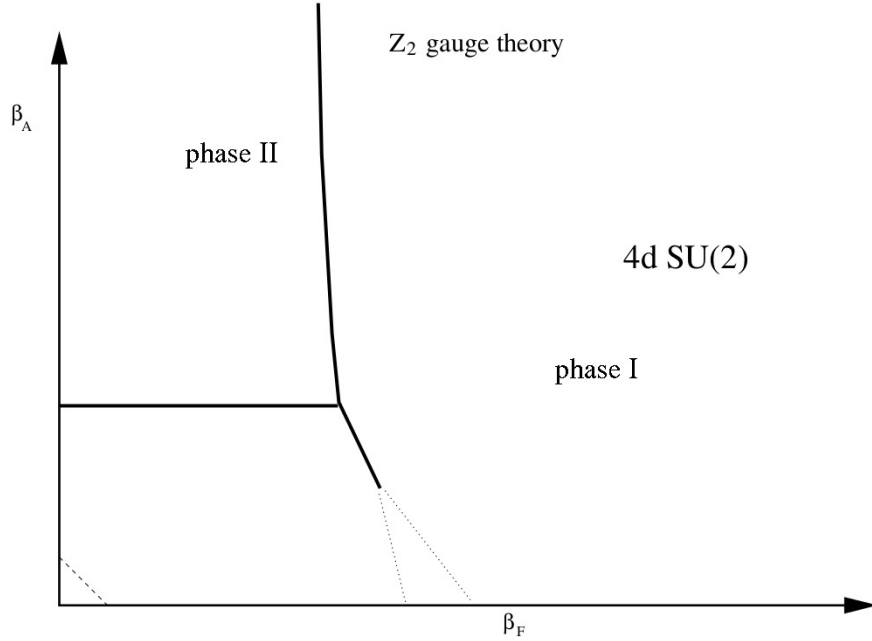


FIG. 3. Phase diagram of the fundamental-adjoint plane for $d = 4$. Continuous lines indicate bulk transitions, dashed lines the roughening transition, dotted lines the crossover regions associated with \mathbb{Z}_2 monopoles. Similar diagrams hold for higher N .

theory has no phase transition; apart from the roughening transition [48], the corresponding phase diagram should therefore be free of any bulk effects, including crossovers.

From Fig. 3 it is obvious that two distinct continuum limits in $d = 4$ exist, depending if $g^2 \rightarrow 0$ in Eq. (7) is taken within Phase I or II. Phase II at fixed twist has been shown in Refs. [27, 29] to be equivalent to a positive plaquette model [14, 55, 56] with fixed twisted boundary conditions. Although such model and the fundamental Wilson action seem to describe the same physics, the two phases are always separated by a bulk transition line.¹⁰ What is thus the difference, if any, between them?

A first hint towards an explanation to this (long neglected) puzzle is given by the results of Refs. [21, 22, 29, 47, 57–62]: in the continuum limit the $d = 4$ adjoint theory ($\beta_F \equiv 0$), which lies precisely within phase II, possesses well defined \mathbb{Z}_2 topological sectors, i.e. no open vortices: the Hilbert-space decomposition defined in Eq. (2) works! On the other hand, one can easily check that in phase I, across all crossovers, the \mathbb{Z}_2 monopole density vanishes quite slowly as $g^2 \rightarrow 0$: their persistence in the weak coupling phase should reflect itself in the presence of open \mathbb{Z}_2 vortices, possibly spoiling Eq. (2). Could the difference between phase I and II lie in whether such super-selection rule is indeed realized for the Hilbert space of Yang-Mills theories? To find out, we can start from the twist operator, which “counts” the \mathbb{Z}_2 vortices piercing all parallel planes for a fixed choice of μ - ν [13, 27]:

$$z_{\mu\nu} = \frac{1}{L^{d-2}} \sum_{\hat{y} \perp \mu\nu\text{-plane}} \prod_{\hat{x} \in \mu\nu\text{-plane}} \text{sign}(\text{Tr}_F U_{\mu\nu}(\hat{x}, \hat{y})). \quad (9)$$

$U_{\mu\nu}$ and \hat{x} , respectively, denote a 1×1 plaquette and point lying in the μ - ν plane, while \hat{y} denotes a point on its co-set, which is obviously empty in $d = 2$; only a single plane contributes to the sum in this case. Notice how $z_{\mu\nu}$, like σ_c , is unaffected by any multiplication of links by a center element, i.e. it is insensitive to the spurious \mathbb{Z}_2 gauge degrees of freedom.

If topological sectors are well defined, all parallel planes will contribute with the same sign to the sum in Eq. (9). For any fixed μ and ν , $z_{\mu\nu}$ can thus only take the values ± 1 , depending on the boundary conditions chosen.¹¹ E.g., for the periodic boundary conditions considered in this paper, the topological sector must always be trivial: $z_{\mu\nu} \equiv 1 \ \forall \mu, \nu$. When, however, topological sectors are ill defined the contributions to the sum in Eq. (9)

¹⁰ The authors of Ref. [27] also proved that the 1st order line separating the two phases is just a finite volume effect: at high enough volume Phase I and II will be always separated by a 2nd order line.

¹¹ Only for $\beta_F = 0$, i.e. along the β_A axis, the $z_{\mu\nu}$ are allowed to tunnel among different topological sectors, provided that an ergodic algorithm capable of overcoming the large barriers among them is used. In this case the $z_{\mu\nu}$ can take both values ± 1 [21, 62].

can change from plane to plane; in particular, if open \mathbb{Z}_2 vortices pierce the planes randomly, all $z_{\mu\nu}$ will average to zero. To make such statement quantitative and characterize how the transition from the disordered to the ordered regime takes place we define a (non-local!) order parameter, the center flux z , such that its expectation value $\langle z \rangle \equiv 1$ if, whatever the boundary conditions, vortex topology takes the correct value expected from the superselection rule, while $\langle z \rangle \equiv 0$ when \mathbb{Z}_2 fluxes are maximally randomized. For $d \geq 3$:

$$z = \frac{2}{d(d-1)} \sum_{\mu > \nu = 1}^d |z_{\mu\nu}|, \quad (10)$$

while for $d = 2$, since $|z_{12}| \equiv 1$, we will define:

$$z = 1 - |z_{12} - \langle z_{12} \rangle|. \quad (11)$$

Notice that the latter definition will only work as long as $\beta_F \neq 0$, i.e. when the $d = 2$ theory cannot tunnel among topological sectors.¹² In the following we will investigate, either analytically (in $d = 2$) or via Monte-Carlo simulations (for $d \geq 3$), the behaviour of the center flux and its susceptibility:¹³

$$\chi_z = L^d (z - \langle z \rangle)^2. \quad (12)$$

B. Algorithm

Simulations for $\beta_A = 0$, i.e. along the β_F axis, have been performed using a standard heat-bath algorithm followed by micro-canonical steps. Although this cannot be extended to $\beta_A \neq 0$, as long as also $\beta_F \neq 0$ one can use the biased Metropolis + micro-canonical algorithm introduced in Refs. [63, 64].¹⁴ The lookup tables for the pseudo-heat-bath probability need to be fixed beforehand: sizes between 32×32 and 64×64 were found to be sufficient [63, 64]. As long as $\beta_F \gg \beta_A$, the algorithm is for all practical purposes just as efficient as an heat-bath,

¹² The definition of the center flux in $d = 2$ might also be adjusted to the pure adjoint theory as long as no ergodic algorithm is available in the ordered phase. The issue is similar to that encountered for e.g. an Ising model when simulating the low-temperature phase with a cluster algorithm.

¹³ Since z is non-local, one could argue that the volume factor should be substituted by the number of planes $\frac{d(d-1)}{2} L^{d-2}$. This would however just change the critical exponent for χ_z from $L^\gamma \rightarrow L^{\gamma-2}$, which could be re-absorbed in the definition of the hyper-scaling relations. Moreover for each plane up to L^2 vortices can form, summing up again to L^d . To underline the analogies of our results with the Kosterlitz-Thouless literature we will thus stick to the standard definition. Anyhow, critical behaviours are controlled by a diverging correlation length ξ , which remains unaffected by any re-scaling of χ_z .

¹⁴ See e.g. Ref. [65] for a recent application. A similar algorithm had been proposed in Ref. [66] for $SU(3)$.

as the amount of accepted proposals stays well above 95%. On the other hand, whenever $\beta_F \ll \beta_A$ the rejected pseudo-heat-bath and micro-canonical updates increase considerably. This becomes a real issue when simulating around the peaks of the susceptibility Eq. (12), where auto-correlations for z and χ_z become quite large.¹⁵ One can try to combat such critical slowing down,¹⁶ unavoidable when dealing with any phase transition, by increasing the number of micro-canonical steps per biased Metropolis update. Unfortunately this turns out to be less efficient than for the $\beta_F \gtrsim \beta_A$ case or for the heat-bath algorithm; only with runs of order $\sim 10^8$ sweeps one eventually reaches a good signal-to-noise ratio for χ_z . Since the $d = 3$ case will anyway turn out to be the most interesting from the point of view of the critical behaviour, while in $d = 2$ analytic results allow to gain otherwise control of the problem, we have limited a precise finite size scaling (FSS) [67] analysis to determine the properties of the transition to the $\beta_A = 0$, $d = 3$ case. Still, we have performed simulations for a whole range of parameters and lattice sizes L in $2 \leq d \leq 4$, trying to explore the whole $\beta_F - \beta_A$ plane. We have nevertheless avoided phase II of the $d = 4$ phase diagram in Fig. 3, since it would have called for completely different simulation techniques; see Refs. [21, 22, 29, 47, 57–62] for results in this parameter region.

III. RESULTS

A. $d = 2$

The $SU(2)$ theory in $d = 2$ offers the chance to tackle our problem analytically [68, 69]. The probability distribution for the mixed action in Eq. (7) reads:

$$d\rho(\theta, \beta_F, \beta_A) \propto d\theta \sin^2 \theta e^{\beta_F \cos \theta + \frac{4}{3} \beta_A \cos^2 \theta}, \quad (13)$$

so that the probability for a plaquette to have negative trace is simply given by:

$$p(\beta_F, \beta_A) = \frac{\int_{\frac{\pi}{2}}^{\pi} d\theta \sin^2 \theta e^{\beta_F \cos \theta + \frac{4}{3} \beta_A \cos^2 \theta}}{\int_0^{\pi} d\theta \sin^2 \theta e^{\beta_F \cos \theta + \frac{4}{3} \beta_A \cos^2 \theta}}. \quad (14)$$

¹⁵ Other observables remain, on the other hand, mostly unaffected.

¹⁶ The critical slowing down appears of course also in the limit $g^2 \rightarrow 0$, i.e. for large β_F and/or β_A .

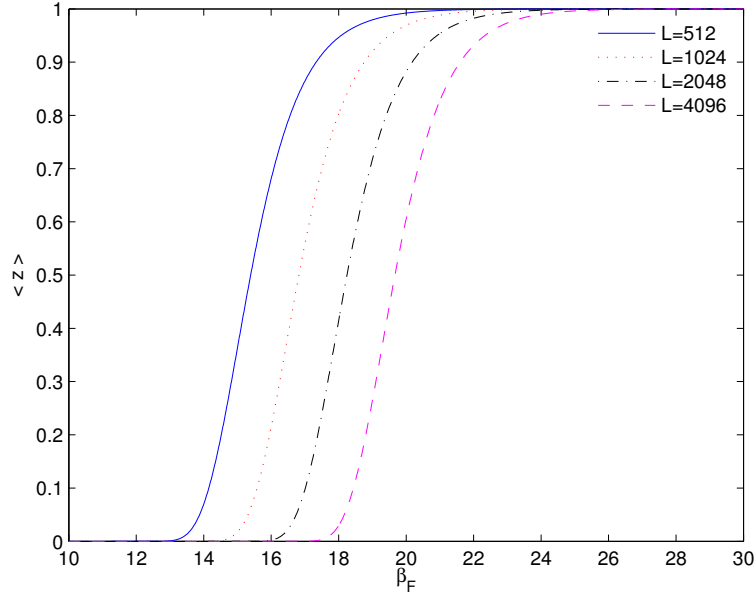


FIG. 4. Order parameter $\langle z \rangle$ in $d = 2$ along β_F for $L = 512, 1024, 2048$ and 4096 .

The limiting cases $\beta_{F,A} \rightarrow 0, \infty$ can be carried out explicitly, giving:

$$\begin{aligned} p(\beta_F \equiv 0, \beta_A) &= \frac{1}{2} \\ p(\beta_F \equiv \infty, \beta_A) &= 0 \end{aligned} \tag{15}$$

$$\begin{aligned} p(\beta_F, \beta_A \equiv 0) &= \frac{1}{2} \left[1 - \frac{L_1(\beta_F)}{I_1(\beta_F)} \right] \\ p(\beta_F, \beta_A \equiv \infty) &= \frac{1}{1 + e^{2\beta_F}}, \end{aligned} \tag{16}$$

where L and I denote the modified Struve and Bessel functions, respectively [70].

For fixed volume L^2 the order parameter z and its susceptibility χ_z are given by [71]:¹⁷

$$\begin{aligned} \langle z \rangle &= e^{-4L^2 p(\beta_F, \beta_A)} \\ \langle \chi_z \rangle &= L^2 \left[e^{-4L^2 p(\beta_F, \beta_A)} - e^{-8L^2 p(\beta_F, \beta_A)} \right]. \end{aligned} \tag{17}$$

The above expressions are plotted, for $\beta_A = 0$, in Fig. 4 and 5; a similar behaviour extends to the whole (β_F, β_A) plane, see Fig. 6, where the center flux is plotted for fixed $L = 128$. We can clearly distinguish a low β_F , “strong” coupling regime, where $\langle z \rangle = 0$ and the topology

¹⁷ We are indebted to F. Bursa for precious correspondence on the derivation of the above expressions for z and χ_z . Just to be on the safe side, we have also cross-checked all analytic results with Monte-Carlo simulations up to $L = 1024$; these become of course inefficient as β_A gets large...

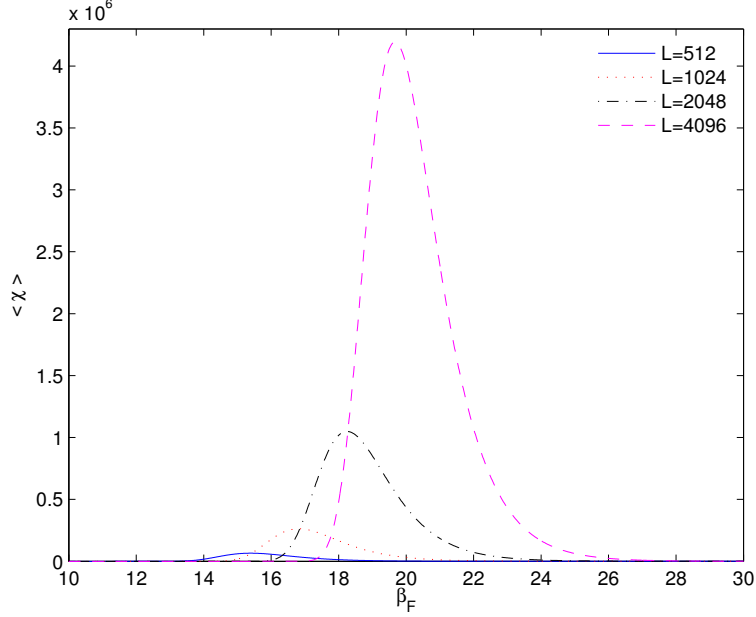


FIG. 5. Susceptibility χ_z in $d = 2$ along β_F for $L = 512, 1024, 2048$ and 4096 .

is ill defined, from a high β_F , “weak” coupling one, where $\langle z \rangle = 1$, the correct value it should have if the vacuum satisfies Eq. (4). For higher L the transition “front” simply moves to the right, i.e. higher β_F ; see Fig. 7, where the curves along which the susceptibility χ_z peaks are plotted for $L = 64, 256, 1024$ and 4096 .

As usual in a FSS analysis, we can determine the properties of the transition by defining the pseudo-critical couplings $(\beta_F^c(L), \beta_A^c(L))$ at finite L as those for which the correlation length $\xi \simeq L$ [67]. These can be identified through the peaks of the susceptibility χ_z (see Fig. 7); since $p(\beta_F, \beta_A)$ has no stationary points, from Eq. (17) one simply needs to solve:

$$p(\beta_F^c, \beta_A^c) = \frac{\log 2}{4 L^2}. \quad (18)$$

Substituting the above value into Eq. (17) we get for the scaling of the center flux and its susceptibility with L :

$$\begin{aligned} z(\beta_F^c, \beta_A^c) &= \frac{1}{2} \\ \chi_z(\beta_F^c, \beta_A^c) &= \frac{L^2}{4}. \end{aligned} \quad (19)$$

As for the scaling of the pseudo-critical points with L , from Eqs. (15) we have that along

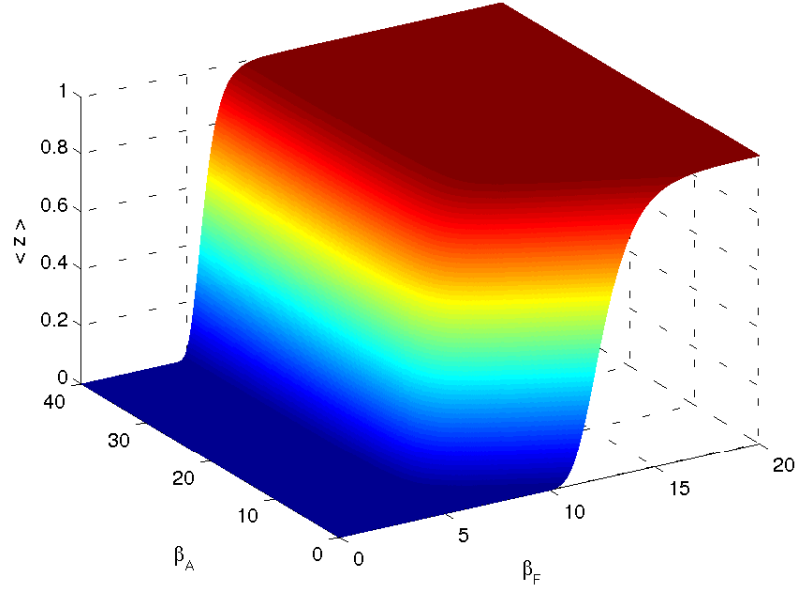


FIG. 6. Order parameter $\langle z \rangle$ in $d = 2$ for fixed $L = 128$.

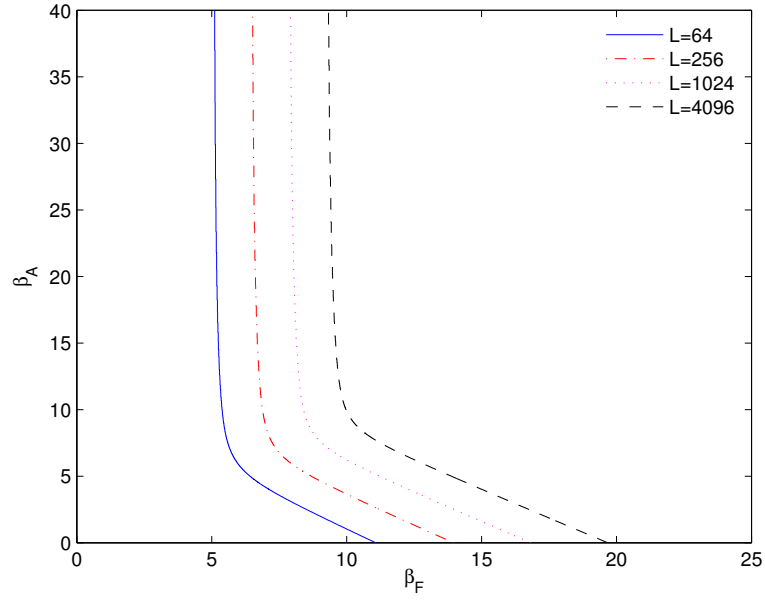


FIG. 7. Peak curves of the susceptibility χ_z in the $\beta_F - \beta_A$ plane for $L = 64, 256, 1024$ and 4096 .

lines parallel to the β_A axis $\langle z \rangle = 0$. Otherwise, we can fix a line $\beta_A = f(\beta_F)$ and solve:

$$p(\beta_F, f(\beta_F)) = \frac{\log 2}{4 L^2} \quad (20)$$

for β_F . In particular, from Eq. (16) and using the asymptotic expansion [70]:

$$p(\beta_F, \beta_A \equiv 0) \sim \sqrt{\frac{2\beta_F}{\pi}} e^{-\beta_F} \left[1 + \mathcal{O}\left(\frac{1}{\beta_F}\right) \right] \quad (21)$$

we get for the two limiting cases $\beta_A \rightarrow 0, \infty$:

$$\beta_F^c(L)|_{\beta_A=0} \sim \log L^2 + \frac{1}{2} \log \log L^2 + \mathcal{O}(1) \quad (22)$$

$$\beta_F^c(L)|_{\beta_A=\infty} \sim \frac{1}{2} \log L^2 + \mathcal{O}(1) : \quad (23)$$

shifting Eqs. (17) by Eqs. (22, 23) $\langle z \rangle$ and $\langle \chi_z \rangle$ will fall on top of each other.

Inverting Eqs. (22, 23), one can extract the critical behaviour of the correlation length $\xi \sim L$ for $\beta_A \rightarrow 0, \infty$, where the pre-factors come from the $\mathcal{O}(1)$ terms:

$$\xi|_{\beta_A=0} \sim \sqrt[4]{\frac{\pi \log^2 2}{2\beta_F}} \cdot e^{\frac{1}{2}\beta_F} \quad (24)$$

$$\xi|_{\beta_A=\infty} \sim \sqrt{\frac{\log 2}{4}} \cdot e^{\beta_F}. \quad (25)$$

Similar expressions will hold for any direction $\beta_A = f(\beta_F)$ along which the continuum limit $\beta_F \rightarrow \infty$ is taken.¹⁸ At $T > 0$ one can simply substitute $L^2 \rightarrow L_s \cdot L_t$ in Eqs. (18, 19, 22, 23). The scaling behaviour remains thus, up to a factor, unchanged when taking the thermodynamic limit $L_s \rightarrow \infty$: the critical behaviour will persist for any fixed L_t , i.e. at any temperature.

Compare now the above scaling with the critical behaviour of the Kosterlitz-Thouless universality class [72–74] as a function of the reduced coupling β_{red} :

$$\xi_{KT} \sim K e^{A \beta_{\text{red}}^\nu} \quad (26)$$

$$\beta_{\text{red}}^{-1} = |\beta^{-1} - \beta_c^{-1}| \propto |T - T_c|. \quad (27)$$

Albeit with a different critical exponent, $\nu = 1$ and $\nu = 1/2$ respectively, both cases show essential scaling, i.e. the correlation length diverges exponentially as one approaches the critical coupling, which in our case is $\beta_F^c = \infty$. Mimicking now a well-known argument

¹⁸ Of course, this only holds as long as $f(\beta_F) \sim \beta_F$ for large β_F .

[73, 74], we can give a simple explanation for the behaviour found in Eqs. (24, 25). At weak coupling the free energy cost to change the sign of a plaquette is $f \sim 2\beta_F$; the density of negative plaquettes will thus be controlled by a Boltzmann factor $\rho \sim \exp(-f)$. On the other hand the possible positions for this sign flip will scale like L^2 and the balance between free energy and entropy gives $L \sim \rho^{-1/2} = \exp(\beta_f) \simeq \xi$.¹⁹ Up to the power correction for $\beta_A = 0$ case, Eq. (24), this simple argument works quite well, contrary to the XY -model, where it cannot explain renormalization effects leading to the non trivial critical exponent $\nu = 1/2$. Moreover, since the minimal distance among vortices can be reliably estimated with that along a plane intersecting them, such picture should (roughly) hold in higher dimensions as well.

We could in principle explore the similarities with the Kosterlitz-Thouless transitions further. Although, as far as we know, for the XY -model no local order parameter is available, one can couple the theory to an external magnetic field h and study the analytical continuation of the partition function $Z(\beta, h)$ to the complex plane. (Hyper-) scaling relations will then hold among the critical exponents of ξ , of the magnetic susceptibility $\chi_h \sim \xi^{2-\eta} \log^{-2r} \xi$, of the specific heat C_s and of the edge of the Lee-Yang zeroes [72]. We will avoid such a throughout analysis in our case, for which a dedicated paper would be needed. Let us however just briefly comment on two points. First, from Eq. (13) we can explicitly calculate the reduced partition function and the specific heat in our usual limiting cases:

$$Z \underset{\beta_A \rightarrow 0}{\propto} \frac{1}{\beta_F} I_1(\beta_F) \quad (28)$$

$$Z \underset{\beta_A \rightarrow \infty}{\propto} \frac{e^{\beta_A}}{\sqrt{\beta_A^3}} \cosh \beta_F \quad (29)$$

$$C_s \underset{\beta_A \rightarrow 0}{=} \frac{3}{2\beta_F^2} \quad (30)$$

$$C_s \underset{\beta_A \rightarrow \infty}{=} 1 - \tanh^2 \beta_F. \quad (31)$$

Inserting Eqs. (22, 23) into Eqs. (30, 31) and assuming that no other contribution besides the singular one exist [72], we see that the critical behaviour for C_s should change (continuously?) from $\log^{-2} L$ to L^{-2} . Second, in our case we have direct access to a non local, topological order parameter, for which we can determine a critical exponent, $z \sim M L^{-\beta}$; from Eq. (19) we have $\beta = 0$. If we would like to study the extended partition function $Z(\beta_F, \beta_A, h)$ we

¹⁹ We wish to thank P. de Forcrand for useful comments about this point.

could simply add a term $S_h = h z$ to the action. Although a direct calculation would go beyond the scope of this paper, it is obvious that for fixed L a sufficient condition to align the center flux z is realized if $\beta_F \rightarrow \infty$: the fundamental coupling plays the role of a "mock" \mathbb{Z}_2 magnetic field. Indeed, from Eqs. (28, 29) the zeros of Z in the complex β_F plane all lie on the imaginary axis, in agreement with the Lee-Yang theorem [75].

Let us finally turn to the continuum limit. From the above discussion it is clear that taking the thermodynamic limit $L \rightarrow \infty$ before the weak coupling limit $\beta_F \rightarrow \infty$, as one should, i.e. taking the Euclidean volume $V = (a L)^2 \rightarrow \infty$ (or, at finite temperature, $V_s = (a L_s) \rightarrow \infty$), the theory remains stuck in the disordered phase $\langle z \rangle = 0$: no vortex topological sector can be defined and the super-selection rule of Eq. (2) is not realized.

On the other hand, assuming that the scaling of the string tension σ with the lattice spacing a , known analytically for $\beta_A = 0$:

$$\begin{aligned}\beta_F &= \frac{4}{a^2 g^2} \\ \sigma &= \frac{3}{8} g^2,\end{aligned}\tag{32}$$

will hold up to a different prefactor along any line $f(\beta_F) \propto \beta_F$, we get:

$$V = (a L)^2 = \frac{3}{2 \sigma \beta_F} L^2.\tag{33}$$

Keeping now the volume V fixed as the continuum limit is approached, the values of the coupling at which one needs to simulate for fixed L will scale as $\beta_F \sim L^2$, i.e. much higher than the pseudo-critical coupling $\beta_F^c \sim \log L$. The theory will thus be in a pseudo-ordered phase with $\langle z \rangle = 1$: on a *finite* Euclidean $d = 2$ torus the Wilson action can admit well defined \mathbb{Z}_2 topological sectors.²⁰

B. $d = 3$

Increasing the dimensions to $d = 3$ we expect interactions to arise among parallel planes, since vortices are now extended, one-dimensional objects. The simple picture we have found

²⁰ Some interpretation issues of course arise in this case. E.g., speaking of zero temperature for a compactified, periodic time is at best misleading. Of course, one could also consider the case $L_s^2 \sim \beta_F$, but to fix the temperature independently one must resort to an anisotropic (Hamiltonian) setup [76]. Transitions on finite toruses in the large N limit of the $d = 2$ Yang-Mills theories have been the subject of intense investigations; see Refs. [77, 78] and references therein.

	$L = 24$	$L = 32$	$L = 40$	$L = 48$	$L = 64$
$\beta_F^c(L)$	5.61(1)	5.930(5)	6.174(2)	6.383(5)	6.700(5)
$\chi_z(\beta_F^c(L))$	152.90(4)	282.83(8)	454.24(12)	666.6(2)	1217.4(3)
$z(\beta_F^c(L))$	0.392(3)	0.3610(15)	0.3364(13)	0.3213(16)	0.2908(16)
$C_s(\beta_F^c(L))$	0.1078(3)	0.0957(1)	0.0877(2)	0.0817(3)	0.0739(4)
$\delta\beta_F$	0.025	0.0125	0.00625	0.003125	0.0015625

TABLE I. Position and height of the susceptibility peaks along β_F in $d = 3$. The third and fourth line give the values of the order parameter and of the specific heat at the pseudo-critical point $\beta_F^c(L)$; the last line gives the coupling steps for the simulations used in the re-weighting.

in $d = 2$ won't probably work anymore and less trivial critical exponents might arise. Still, fluxes are inherently two-dimensional objects and most of the dynamics should thus take place on planes: many features of the $d = 2$ case should therefore survive. To check this, we have performed sets of Monte-Carlo simulations along different lines in the $\beta_F - \beta_A$ plane. Results are reported for $\beta_A = 0$, $\beta_F = 0.5, 0.75$ and lattice sizes between $L = 24$ and $L = 80$; other parameters have been checked and give a consistent picture.

In the $\beta_A = 0$ case approximately 20 to 50 simulations at coupling steps $\delta\beta_F$, each with 10^6 independent configurations, were performed for each volume L^3 . The data have been re-weighted [79, 80] to determine the peak values $\beta_F^c(L)$ and $\chi_z(\beta_F^c(L))$; this was viable only up to $L \sim 64$. Indeed, as we shall see below, the $d = 3$ case shows a similar scaling behaviour as Eq. (22), i.e. a logarithmic scaling of $\beta_F^c(L)$ to a critical coupling $\beta_F^c = \infty$. This has a practical drawback: the absolute width of the transition, i.e. the overall interval $\Delta\beta_F$ one needs to simulate, varies very slowly, while the step-width $\delta\beta_F$ one must scan in order to keep the density of states computationally feasible decreases dramatically with L : the computational cost becomes eventually unmanageable.

Results for all volumes considered are resumed in Tab. I, where the steps $\delta\beta_F$ are also listed, along with the value of the center flux and, for sake of completeness, of the specific heat at the pseudo-critical point. To cross-check scaling results, similar simulation steps and statistics have also been used for the other volumes not included in the re-weighting. The

data can be well fitted with the Ansatz:

$$\chi_z(\beta_F^c(L)) \sim A L^{2-\eta} \log^{-2r} L (1 + \mathcal{O}(L^{-1})) \quad (34)$$

$$z(\beta_F^c(L)) \sim M L^{-\beta} (1 + \mathcal{O}(L^{-1})) \quad (35)$$

$$\beta_F^c(L) \sim C \log L^2 + D \log \log L^2 + \mathcal{O}(1), \quad (36)$$

For χ_z we get $A = 0.21(1)$, $r = -0.134(10)$, $\eta = 0.0001(100)$ and $\chi^2/\text{d.o.f.} = 5.7$; constraining $\eta = 0$ gives again $A = 0.21(1)$, $r = -0.134(10)$ with $\chi^2/\text{d.o.f.} = 2.8$. For β_F^c we get $C = 0.61(3)$ and $D = -0.42(5)$ with $\chi^2/\text{d.o.f.} = 0.7$; on the other hand, constraining $D = 0$ we get $C = 0.56(3)$, $\chi^2/\text{d.o.f.} = 0.6$. Finally, for the order parameter, we get $M = 1.26(5)$ and $\beta = 0.35(1)$ with $\chi^2/\text{d.o.f.} = 1.4$. Overall, the biggest source of systematic error is given by the parameterization of the sub-leading corrections: leaving them out or parameterizing them differently leads to changes of up to 10% for some of the critical exponents, not included in our error estimates; obviously, more data at higher volumes are needed to pin the numbers down.²¹ The data for the susceptibility χ_z , rescaled by Eqs. (34, 36), are plotted in Fig. 8, showing very good agreement also for the volumes which have not been included in the re-weighting analysis. In Fig. 9 we show the scaling of the order parameter z according to Eqs. (35, 36); the agreement for $L \gtrsim 40$ is again very good. As for the specific heat, a fit of the data in Tab. I with a logarithmic Ansatz $C_s \sim \log^{-\alpha} L$ gives $\alpha = 1.4(1)$ with a $\chi^2/\text{d.o.f.} = 0.5$. The signal-to-noise ratio for the MC is however not so good in this case, reflecting itself in the quality of the re-weighted data: more statistics would be definitely needed; anyway, checking any (hyper-) scaling relation is beyond our goals.

The above result is quite surprising. Indeed, in contrast to $d = 2$, one could have expected the \mathbb{Z}_2 monopole to control the open center vortices, since the density of the latter is proportional to that of the former. However, although monopoles per unit volume steadily decrease beyond the cross-over, open vortices “connecting” them still cause a critical behaviour cumulating to $g^2 \rightarrow 0$.²² A possible explanation could be that their length increases more than linearly with the lattice size; multiple bendings in orthogonal directions would be enough to randomize the fluxes. A direct investigation of any geometrical properties of

²¹ Another possible issue could be the non-ergodicity of our set-up in the ordered phase. Indeed, a “good” algorithm would need to change boundary conditions to enable tunneling among different topological sectors around the transition, just like a cluster algorithm in an Ising model allows tunneling among different orientations of the spins in the spontaneously magnetized phase. See e.g. Ref. [23] for possible solutions to the problem. Implementing such algorithm is obviously beyond the scope of this paper.

²² A somewhat cryptic comment regarding a possible critical behaviour, going as far as taking the XY-model as a paradigm, can be found in Ref. [37].

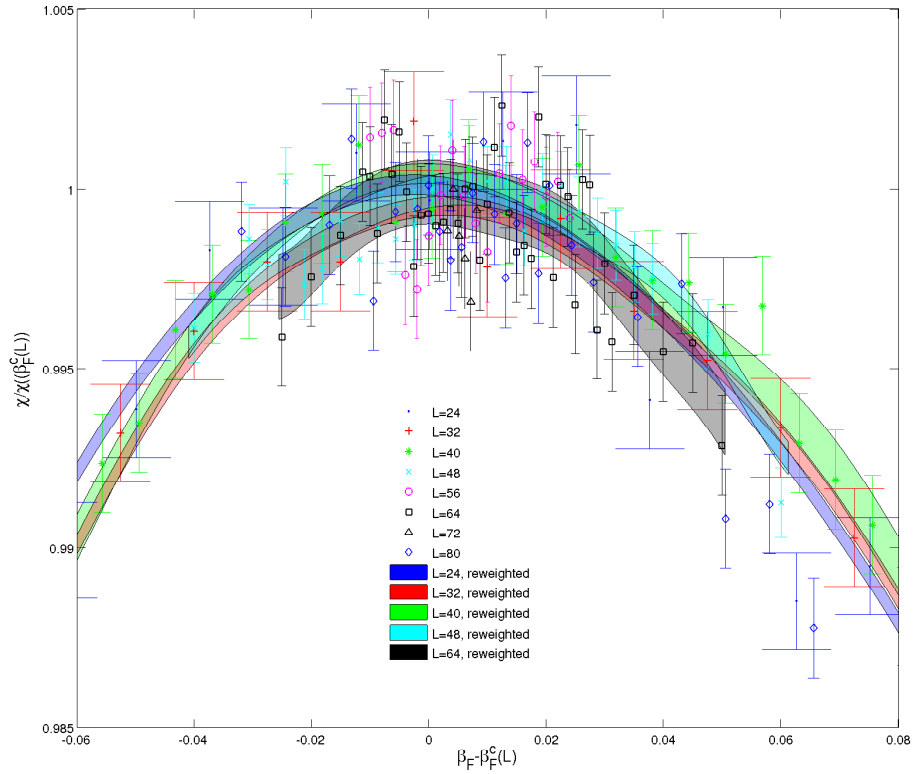


FIG. 8. Data for the susceptibility of the order parameter in $d = 3$, including the re-weighted curves, rescaled with the FSS Ansatz in Eqs. (34, 36).

open vortices is however beyond the scope of this paper, since Eq. (9) is non local and gauge invariant and does not allow to isolate the topological defects on the planes.

Going now to the $\beta_A \neq 0$ case, since the \mathbb{Z}_2 monopoles undergo a cross-over also in the low β_F region of the phase diagram of Fig. 2, one would expect the center flux to behave as in the β_F case: one should find along β_A a similar scaling as in Eqs. (34, 36). Also, the transition lines should not be effected by the bulk transition associated with the unphysical \mathbb{Z}_2 gauge degrees of freedom. However, such strong transition unavoidably makes any simulation near it quite noisy; on top of that the biased Metropolis algorithm, with e.g. 3 micro-canonical steps, gets inefficient as β_F gets small and β_A large, reaching for z and χ_z , around the peaks of the latter, integrated autocorrelation times of the order $10^4 - 10^5$ for $24 \leq L \leq 40$. Passable data were therefore only accessible for three volumes, while gathering enough statistics to re-weight the susceptibility was out of the question. We have

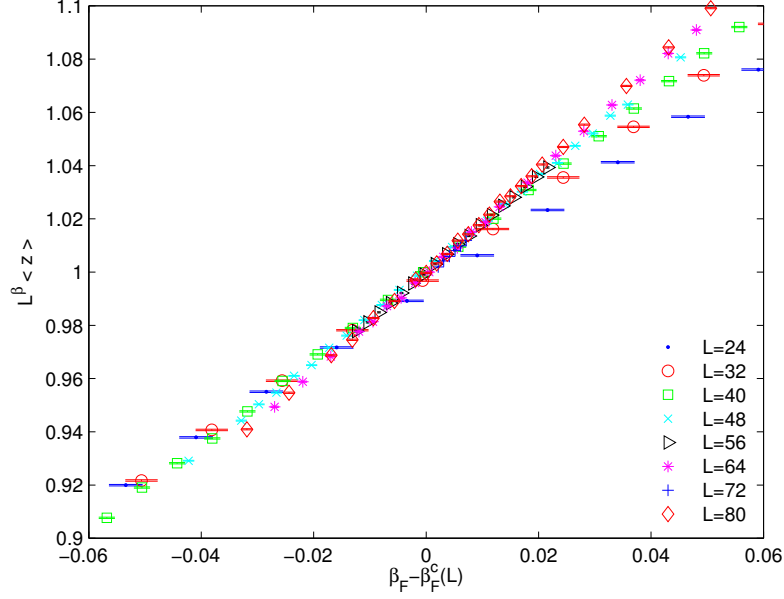


FIG. 9. FSS for the order parameter z as a function of the rescaled coupling as in Eq. (35).

thus limited ourself to a consistency check near the bulk transition with a scaling Ansatz similar to Eqs. (34-36):

$$\chi_z(\beta_A^c(L)) \sim A L^2 \log^{-2r} L + \mathcal{O}(L) \quad (37)$$

$$z(\beta_A^c(L)) \sim M L^{-\beta} (1 + \mathcal{O}(L^{-1})) \quad (38)$$

$$\beta_A^c(L) \sim C \log L^2 + \mathcal{O}(1); \quad (39)$$

no fit has been attempted. The scaling of the pseudo critical point and of the order parameter are consistent with the $d = 2$, $\beta_A = \infty$ case, $C = 1/2$ and $\beta = 0$, as can be seen from Fig. 10. On the other hand, the peaks of χ_z are quite noisy and even a consistency check for the logarithmic exponent is hopeless. In Fig. 11, 12 we show the results for the simulations along $\beta_F = 0.5$ and $\beta_F = 0.75$, lying respectively left and right of the bulk transition, rescaled by Eq. (37, 39) with a “guessed” value for $r = -1/2$; of course, further work would be needed to determine the critical exponents reliably.

We have also checked via Monte-Carlo simulations that all of the above results generalize to $T > 0$ by simply substituting $L^2 \rightarrow L_s \cdot L_t$ in all the scaling relations for temporal fluxes, while the behaviour of all spacial fluxes remains unchanged. Again, as in the $d = 2$ case, this implies that, along any line in the $\beta_F - \beta_A$ plane, when taking the thermodynamic limit

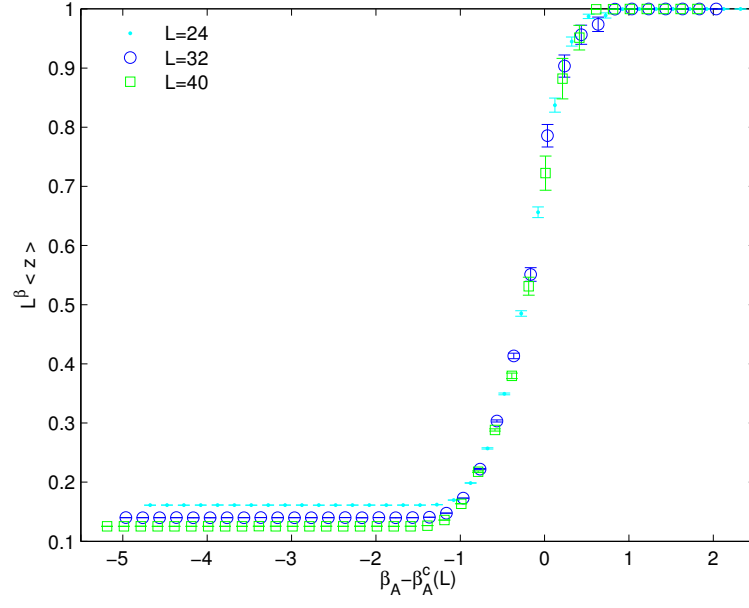


FIG. 10. FSS for the order parameter z along the $\beta_F = 0.5$ line with the Ansatz in Eqs. (37, 38).

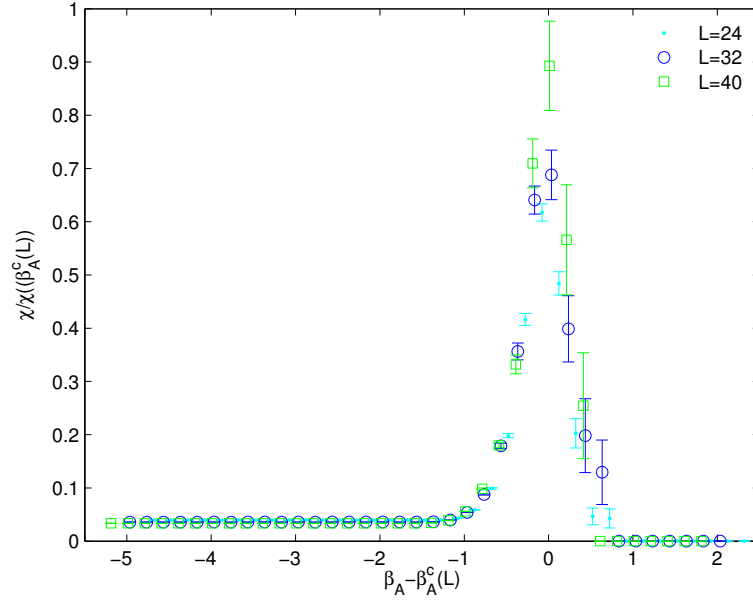


FIG. 11. FSS for the susceptibility χ_z along the $\beta_F = 0.5$ line with the Ansatz Eqs. (37, 39).

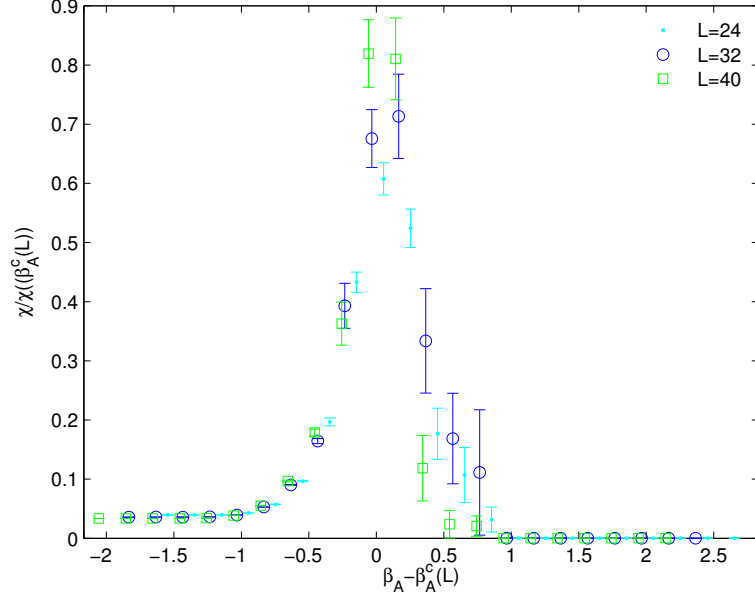


FIG. 12. FSS for the susceptibility χ_z along the $\beta_F = 0.75$ line with the Ansatz Eqs. (37, 39).

before the weak coupling limit, i.e. sending the volume to infinity, the $d = 3$ theory remains stuck in the disordered phase $\langle z \rangle = 0$; again, Eq. (2) is not realized.

What about the fixed volume limit? Taking as a blueprint for the continuum limit along any direction the scaling of the string tension along β_F [81]:

$$\begin{aligned} \beta_F &= \frac{4}{a g^2} \\ a \sqrt{\sigma} &= \frac{c_0}{\beta_F} + \frac{c_1}{\beta_F^2} + \mathcal{O}\left(\frac{1}{\beta_F^3}\right). \end{aligned} \quad (40)$$

we get immediately:

$$V = (a L)^3 \propto \frac{L^3}{\beta_F^3 \sqrt{\sigma^3}}. \quad (41)$$

Keeping again V fixed as the continuum limit is approached, the values of the coupling corresponding to a given L will now scale as $\beta_F \sim L$; again, as in $d = 2$, they will always be much higher than the pseudo-critical coupling $\beta_F^C \sim \log L$ and the Wilson action could admit well defined \mathbb{Z}_2 topological sectors on a *finite* $d = 3$ torus.

	$L = 12$	$L = 16$	$L = 20$	$L = 24$
$\beta_F^c(L)$	3.15(5)	3.40(5)	3.60(5)	3.75(5)
$\chi_z(\beta_F^c(L))$	24(1)	42(1)	66(1)	95(1)

TABLE II. Position and height of the susceptibility peaks along β_F in $d = 4$.

	$L = 16$	$L = 20$	$L = 24$
$\beta_F = 1.0$	1.845(25)	1.923(25)	1.987(25)
$\beta_F = 1.2$	1.695(25)	1.773(25)	1.837(25)
$\beta_F = 1.3$	1.62(1)	1.70(10)	1.76(1)

TABLE III. Position of the susceptibility peaks along β_A in $d = 4$ for $\beta_F = 1.0, 1.2$ and 1.3 ; the heights are all compatible with the results in Tab. II

C. $d = 4$

The positions of the peaks of χ_z , as obtained in the simulations along the $\beta_A = 0$, $\beta_F = 1.0$, $\beta_F = 1.2$ and $\beta_F = 1.3$ lines, all within phase I of Fig. 3, are shown in Tabs. II, III. We have again limited ourselves to a consistency check with a scaling Ansatz of the form:

$$\chi_z(\beta_F^c(L)) \sim A L^2 + \mathcal{O}(L) \quad (42)$$

$$z(\beta_F^c(L)) \sim M L^{-\beta} (1 + \mathcal{O}(L^{-1})) \quad (43)$$

$$\beta_F^c(L) \sim C \log L^2 + \mathcal{O}(1) . \quad (44)$$

Results are shown in Figs. 13, 14 for the order parameter and its susceptibility along the β_F axis; up to the values of C the behaviour along the lines parallel to the β_A axis is basically the same, see e.g. Fig. 15. From the data in Tab. II we can estimate $C = 0.46(3)$, compatible with $1/2$, while for those in Tab. III we get $C = 0.18(3)$; in all cases β is compatible with 0.

Direct simulations at $T > 0$ give again the same scaling with $L^2 \rightarrow L_s \cdot L_t$ for the temporal fluxes, while spacial fluxes remain unchanged. As in the $d = 2$ and $d = 3$ cases, in the thermodynamic limit the theory remains therefore stuck in the disordered phase. Moreover, starting from the 2-loop beta-function, the running of the physical scale with

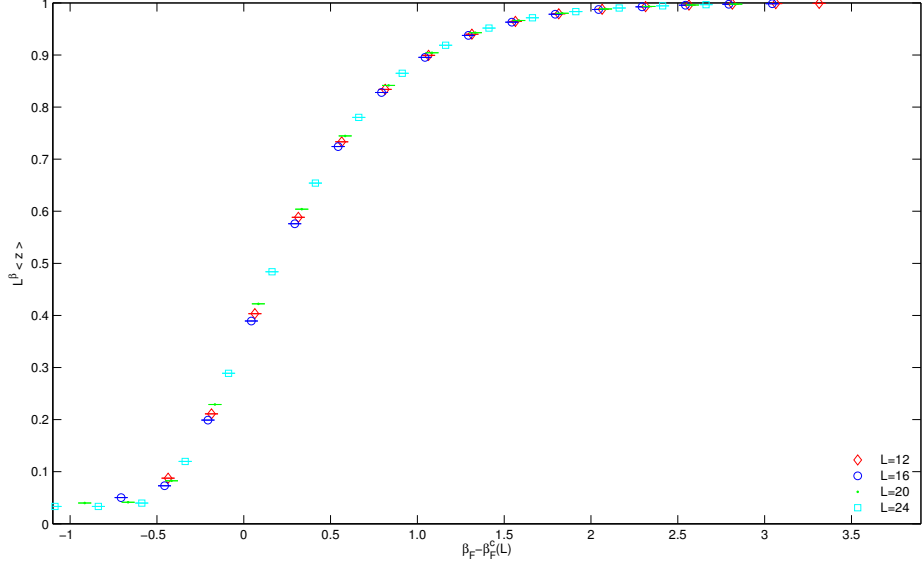


FIG. 13. FSS as in Eqs. (43, 44) for z in $d = 4$ along the β_F axis.

$\alpha_{\text{lat}} = g^2/(4\pi)$ is given by:

$$\log(a^2 \sigma) = -\frac{4\pi}{\beta_0} \alpha_{\text{lat}}^{-1} + \frac{2\beta_1}{\beta_0^2} \log\left(\frac{4\pi}{\beta_0} \alpha_{\text{lat}}^{-1}\right) + c + \mathcal{O}(\alpha_{\text{lat}}), \quad (45)$$

where $c = \log \frac{\sigma}{\Lambda_{\text{lat}}^2}$ and $\beta_0 = \frac{22}{3}$, $\beta_1 = \frac{68}{3}$, i.e. from Eq. (7):

$$V = (aL)^4 \propto L^4 \left(\frac{4\pi^2}{\beta_0} \beta_F \right)^{\frac{4\beta_1}{\beta_0^2}} e^{-\frac{8\pi^2}{\beta_0} \beta_F}. \quad (46)$$

When trying to keep the volume V fixed as $\beta_F \rightarrow 0$, up to log log corrections the coupling should scale as:

$$\beta_F \sim \frac{\beta_0}{4\pi^2} \log L^2; \quad (47)$$

the coefficient C in Eq. (44) is however larger than that coming from the beta-function and the simulation parameters will lie in the disordered phase: topological sectors will always be ill-defined also on a finite torus T^4 . The same holds along the lines parallel to the β_A axis (see Tab. III); in this case, from Eq. (7), the coefficient in Eq. (47) coming from the beta-function Eq. (45) is $3\beta_0/(32\pi^2)$, again smaller than the corresponding value of C .

As discussed in Sec. IIB, we have excluded phase II (see Fig. 3) from the simulations. As mentioned above, vortex topology is however well understood in this case: the results of

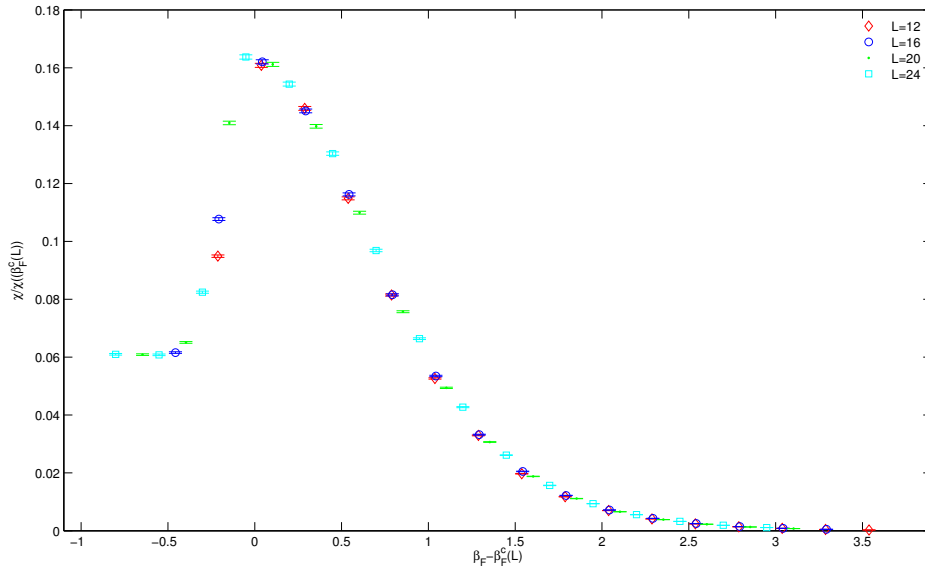


FIG. 14. FSS as in Eqs. (42, 44) for χ_z in $d = 4$ along the β_F axis.

Refs. [21, 22, 29, 47, 60] show that, in contrast to phase I in $d = 4$ and to the $d = 2$ and 3 cases, the theory possesses well defined \mathbb{Z}_2 topological sectors in the continuum limit.

IV. CONCLUSIONS

We have studied a topological order parameter, the center flux z defined in Eqs. (10, 11), for the $SU(2)$ mixed action in $2 \leq d \leq 4$. Its ordered phase, $\langle z \rangle = 1$, corresponds to well defined $\pi_1(SO(3)) = \mathbb{Z}_2$ topological sectors, i.e. to a vacuum satisfying the superselection rule of Eqs. (2, 4), while for $\langle z \rangle = 0$ the vacuum state is disordered and no center topology can be defined. This reminds of a quantum phase transition; however, one does not switch between vacua by tuning a physical parameter. Rather, the choice of dimensions and the symmetry of the discretized action control in which phase the theory will be in the continuum limit.

More specifically, discretized actions transforming in the fundamental representation possess a disordered vacuum, with z showing an essential scaling to the critical coupling $\beta_c = \infty$. The critical exponent for the correlation length ξ is $\nu = 1$, i.e. $\beta_c(\xi) \propto \log \xi$; explicit $\log \log \xi$ corrections to scaling can be shown to exist for some choice of parameters. The susceptibil-

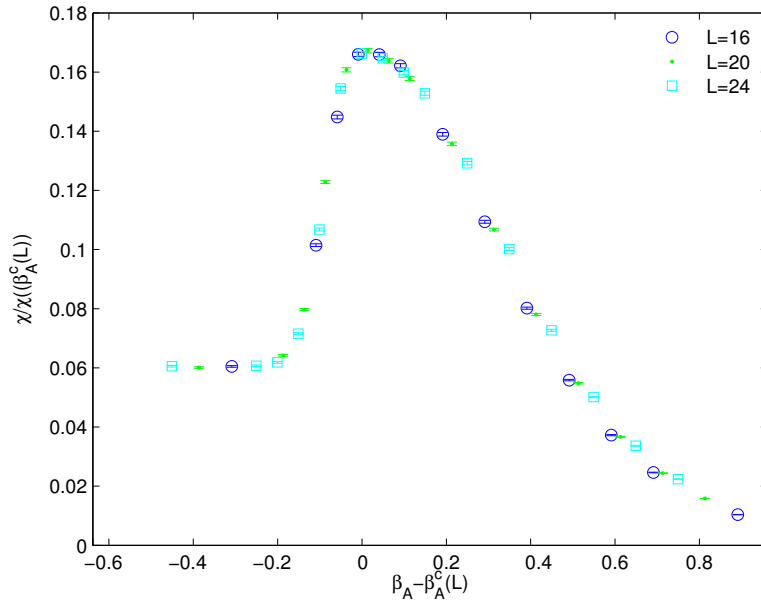


FIG. 15. FSS as in Eqs. (42, 44) for χ_z in $d = 4$ along the $\beta_F = 1.3$ axis.

ity of the center flux scales as $\chi_z(\xi) \propto \xi^2$ in $d = 2$ and $d = 4$, while the order parameter itself scales trivially in these cases. On the other hand in $d = 3$, at least along the β_F axis, the center flux has a non trivial critical exponent, $z(\xi) \propto \xi^{-\beta}$, with $\beta = 0.35(1)$, while a logarithmic correction can be explicitly determined for the scaling of its susceptibility, $\chi_z(\xi) \sim \xi^2 \log^{-2r} \xi$, with $r = -0.134(10)$; similar corrections might also be present along other lines in the $\beta_F - \beta_A$ diagram, but more statistics would be needed to reach a conclusive result. A tentative critical exponent for the specific heat, $C_s(\xi) \sim \log^{-\alpha} \xi$, gives $\alpha = 1.4(1)$, but with still high systematic errors. We have made no attempt to investigate any (hyper-) scaling relations among such exponents; this would probably require a full analysis of the Lee-Yang zeros [72]. Such behaviour persists in all dimensions at $T > 0$.

Vice versa, the topological classification of Eq. (1) and thus the super-selection rule of Eqs. (2, 4) can be realized by the vacuum state of lattice actions transforming in the adjoint representation; phase II in $d = 4$ (see Fig. 3) is such an example [21, 22, 29]. Large scale simulations with the adjoint action are hampered by strong finite-volume effects [27, 28, 38]. Therefore, although the techniques used in Refs [21, 22, 29] to tame them could also work in $d = 3$, a more viable alternative, applicable also in $d = 2$, would be to resort to positive plaquette models [14, 55, 56], where topological sectors are always well defined since

the operator given in Eq. (9) takes “by construction” the values dictated by the assigned boundary conditions. Indeed, a one-to-one mapping between configurations in such lattice discretization and those of the adjoint Wilson action with well defined vortex sectors was conjectured in Ref. [27] and explicitly constructed in Ref. [29]. Finally, an ordered vacuum could also be realized for a finite torus in $d = 2, 3$; here one could exploit the power-law scaling of the physical mass with the coupling to define topological sectors when $L \rightarrow \infty$ and $a \rightarrow 0$ with the volume $V = (aL)^d$ kept fixed.

The above findings do not contradict universality, since non perturbatively the equivalence between fundamental and adjoint actions can only hold as long as no lattice artifacts are present [27–29, 36–39], while as we have seen for some discretizations the density of \mathbb{Z}_2 monopoles can not vanish at any finite coupling [37]. Does however such result have any physical consequences? The vacua of the two different phases can be essentially characterized by the type of \mathbb{Z}_2 vortices they can carry:

- i) The ordered phase allows topological center vortices “à la ’t Hooft” [12, 13]: a confinement mechanism based on the super-selection rule of Eqs. (2, 4) can be realized; at finite temperature the change in the vortex free energy as measured via Eq. (3) is thus a valid test to establish how the symmetry is broken in the transition to the deconfined phase [21, 22, 29]. No fundamental fields are allowed in this case [12, 13, 31]; however, adjoint fermions can be easily incorporated in such scenario. It might therefore be interesting to investigate the vacuum properties of the $SU(2)$ gauge theory coupled to adjoint fermions, a popular candidate for infrared conformality [65]. Numerical tests with the adjoint Wilson action or positive plaquette model should be viable.
- ii) The disordered phase is dominated by (one huge, percolating?) open vortices, reminding of the Nielsen-Olesen “spaghetti vacuum” [4]. Such open vortices are not topological according to Eq. (1): Eqs. (2, 4) cannot be applied. One might conjecture some relationship with P-vortices [2, 15, 16], although it is still unclear how to test such hypothesis, since the center flux z is gauge invariant and constructed out of pure $SO(3)$ variables while P-vortices are gauge dependent and built out of the \mathbb{Z}_2 gauge degrees of freedom. Moreover, such open vortices persist at any temperature, not disappearing above T_C . This disordered vacuum is detached from the boundary conditions chosen and is therefore compatible with the presence of fundamental matter fields. Of course, Eq. (3) is ill defined in this case; whether any vortex related order parameter for the confinement-deconfinement phase transition could be

defined remains an open question.

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